

Stability of Differentially Rotating Disks in $f(T)$ Theory

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ABSTRACT

To explain the accelerated expansion of our universe, many dark energy models and modified gravity theories have been proposed so far. It is argued in the literature that they are difficult to be distinguished on the cosmological scales. Therefore, it is well motivated to consider the relevant astrophysical phenomena on (or below) the galactic scales. In this work, we study the stability of self-gravitating differentially rotating galactic disks in $f(T)$ theory, and obtain the local stability criteria in $f(T)$ theory, which are valid for all $f(T)$ theories satisfying $f(T=0)=0$ and $f_T(T=0)\neq 0$, if the adiabatic approximation and the weak field limit are considered. The information of the function $f(T)$ is mainly encoded in the parameter $\alpha \equiv 1/f_T(T=0)$. We find that the local stability criteria in $f(T)$ theory are quite different from the ones in Newtonian gravity, general relativity, and other modified gravity theories such as $f(R)$ theory. We consider that this might be a possible hint to distinguish $f(T)$ theory from general relativity and other modified gravity theories on (or below) the galactic scales.

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I. INTRODUCTION

The discovery of accelerated expansion of our universe from the observations of Type Ia supernovae in 1998 [1] has been one of the most amazing achievements in modern cosmology. The simplest way to explain this fantastic phenomenon is introducing a cosmological constant [2, 3]. However, it is plagued with the fine-tuning and cosmic coincidence problems (see e.g. [2]). So, many dynamical dark energy (DE) models have been proposed, for instance, quintessence [4], phantom [5] and so on. On the other hand, modifying general relativity (GR) [6] on the cosmological scales has also been extensively considered to explain the accelerated expansion of our universe without introducing DE. In fact, there exist many modified gravity theories in the literature, such as $f(R)$ theory [7], $f(T)$ theory [8, 9, 37], massive gravity [10]. In particular, the well-known $f(R)$ theory [7] is constructed by replacing the Einstein-Hilbert Lagrangian (namely Ricci scalar R) with an arbitrary function $f(R)$. Similarly, the so-called $f(T)$ theory [8, 9, 37] is constructed by replacing the ordinary Lagrangian (namely torsion scalar T) in the teleparallel equivalent of general relativity (TEGR) with an arbitrary function $f(T)$. In fact, there are other modified gravity theories in the literature, such as scalar-tensor theory, braneworld model, Galileon gravity, Gauss-Bonnet gravity, etc. We refer to e.g. [6–10, 29, 37] for more details about modified gravity.

Since we have a flood of models of DE and modified gravity in the literature, some authors focus on how to differentiate one model from others. For example, in order to differentiate dynamical DE from cosmological constant, the expansion history of the universe is considered. Caldwell and Linder proposed a so-called $w - w'$ analysis in [11], and then it was extended in e.g. [12]. Another method is the statefinder diagnostic proposed by Starobinsky *et al.* [13]. We refer to e.g. [14] and references therein for some relevant works on $w - w'$ analysis and statefinder diagnostic. However, as is well known (see e.g. [15]), one can always build models sharing a same cosmic expansion history, and hence these models cannot be distinguished by using the expansion history only. Later, it is realized that if the cosmological models share a same cosmic expansion history, they might have different growth histories characterized by the growth function $\hat{\delta}(\hat{z}) \equiv \delta\rho_m/\rho_m$ (namely the matter density contrast as a function of redshift \hat{z}). Therefore, the cosmological models might be distinguished from each other by combining the observations of both the expansion and growth histories (see e.g. [16]). However, this approach has been challenged. For instance, if non-trivial dark energy clustering (e.g. non-vanishing anisotropic stress) is allowed, it is found in e.g. [17, 18] that DE models still cannot be distinguished from modified gravity theories even by using the observations of both the expansion and growth histories. On the other hand, without invoking non-trivial dark energy clustering, it is found in e.g. [19] that the interacting DE models also cannot be distinguished from modified gravity theories. In fact, the work in [19] was further generalized in [20]. It is found that the interacting DE models, modified gravity theories and warm dark matter models are indistinguishable [20]. Therefore, the complementary probes beyond the ones of cosmic expansion history and growth history are required to distinguish various cosmological models.

Notice that in all the arguments of [16–20], it is assumed that the matter density contrast $\hat{\delta}(\hat{z})$ is linear. If this key assumption is invalid, the hope to distinguish various cosmological models still exists. This draws our attention to the observations on the smaller scales. For instance, the galactic scales might be suitable, since the matter density contrast $\hat{\delta}(\hat{z})$ becomes non-linear on (or below) these scales. So, it is well motivated to consider the relevant astrophysical phenomena on (or below) the galactic scales.

In the present work, we are interested to consider the stability of equilibria of galactic disks. We hope this can bring some possible hints to discriminate various modified gravity models. The study of stability of stellar equilibria is one of the most important tasks in galactic dynamics [21], since it is used in many aspects such as the star formation [22]. For many equilibria of stellar systems, a slight perturbation will cause them evolve violently away from their initial states. In other words, many equilibria of stellar systems are unstable. The instabilities considered in this work are caused by cooperative effects, in which a density perturbation gives rise to extra gravitational forces deflecting the stellar orbits in such a way that the original density perturbation is enhanced. In order to relate to observation, the stellar systems considered in this work is differentially rotating disks, because realistic galactic disks are not static and do not rotate uniformly. Besides, the phenomena of instability in disks are strongly influenced by differential rotation. The stability analysis for differentially rotating systems are more difficult than static spherical systems and uniformly rotating systems. Fortunately, Lin and Shu [23] recognized that the structure of spiral arms in a stellar disk could be regarded as a density wave, a wavelike oscillation that propagates through the disk in much the same way that waves propagates through violin strings. They also realized

that the techniques of wave mechanics, which has been developed into the so-called density wave theory, could be applied to deduce the properties of density waves in differentially rotating stellar disks.

To our knowledge, the stability analysis was firstly investigated by Safronov [24] and Toomre [25], respectively. By studying the behavior of the density waves in self-gravitating disks in Newtonian gravity, Toomre [25] derived the so-called dispersion relation which relates the wavenumber to its frequency, and then he got the local stability criterion (known as Toomre's stability criterion in the literature) which can be used to approximately determine whether the system is locally stable to a small axisymmetric perturbation by applying the dispersion relation. It is worth noting that the gravitational potential plays an important role in deriving the dispersion relation and local stability criterion for a given system. When we study the stability in GR, the derived gravitational potential in the weak field limit reduces to the one in Newtonian gravity. However, in the framework of modified gravity theory rather than GR, it is natural to anticipate a different gravitational potential, and then a different dispersion relation as well as a different local stability criterion for a given self-gravitating system. Therefore, the stability of self-gravitating disk might be used to reflect the difference between various gravity theories.

In the literature, the stability of differentially rotating disks were considered in some modified gravity theories, e.g. Modified Newtonian Dynamics (MOND) [26], Moffat's Modified Gravity (MOG) [27, 36], $f(R)$ theory [28], and so on. In the present work, we try to consider the stability of differentially rotating disks in another modified gravity theory, namely $f(T)$ theory [8, 9, 37]. As one of the important modified gravity theories, $f(T)$ theory has some interesting features. For instance, unlike the metric $f(R)$ theory whose equations of motion are 4th order, the equations of motion in $f(T)$ theory are 2nd order. This virtue makes $f(T)$ theory fairly attractive. On the other hand, it is worth noting that $f(T)$ theory can provide a mechanism for realizing bouncing cosmology [48], thereby avoiding the Big Bang singularity. The seminal work of [37] also obtained a cosmological picture without the initial singularity. This is another one of the advantages of $f(T)$ theory. In fact, there are many relevant works on $f(T)$ theories in the literature (e.g. [34, 41–50]). Actually $f(T)$ theory becomes one of the active fields in cosmology, and it received a huge attention from the very beginning, as is shown by the references in [29].

The rest of this paper is organized as follow. In Sec. II, we briefly review the key points of $f(T)$ theory, and then derive the Poisson's equation in the weak field limit. In Sec. III, we consider the behavior of density waves in differentially rotating disks in $f(T)$ theory by using the density wave theory. We obtain the perturbed gravitational potential by considering a small perturbation of the equilibria, and derive the dispersion relations for both gaseous disks and stellar disks. In Sec. IV, we use the dispersion relations to get the corresponding Toomre's local stability criteria for an axisymmetric perturbation in both gaseous disks and stellar disks. Finally, some brief concluding remarks are given in Sec. V.

II. $f(T)$ THEORY

A. Key points of $f(T)$ theory

Here, we briefly review $f(T)$ theory following e.g. [8, 9, 37]. In fact, it is a generalization of TEGR. Teleparallelism uses the Weitzenböck connection [38] (which has torsion but it is curvatureless) to describe spacetime, while GR uses the Levi-Civita affine connection (which has curvature but it is torsionless). In this sense, teleparallelism is a sector of Einstein-Cartan theories [39, 40], which describe gravity by means of a connection having both torsion and curvature. Teleparallel gravity uses a vierbein field $e_a = e_a^\mu \partial_\mu$ as dynamical quantity, where Greek indices μ, ν, \dots , run over 0, 1, 2, 3 on the manifold, and Latin indices a, b, \dots , run over 0, 1, 2, 3 on the tangent space of the manifold. We use the Einstein notation throughout this work. The vierbein field e^a_μ relates to the spacetime metric $g_{\mu\nu}$ through [8, 9, 37]

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \quad (1)$$

where $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$ is the Minkowski metric. The Weitzenböck connection $\Gamma^\lambda_{\mu\nu}$ and the torsion tensor $T^\rho_{\mu\nu}$ are defined by [8, 9, 37]

$$\begin{aligned} \Gamma^\lambda_{\mu\nu} &= e_a^\lambda \partial_\mu e^a_\nu, \\ T^\rho_{\mu\nu} &= \Gamma^\rho_{\nu\mu} - \Gamma^\rho_{\mu\nu} = -e_a^\rho (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu). \end{aligned} \quad (2)$$

Unlike the Levi-Civita connection, the indices μ and ν in Weitzenböck connection are not symmetric, so the torsion tensor is not vanishing. The torsion scalar is given by [8, 9, 37]

$$T = T^\rho{}_{\mu\nu} S_\rho{}^{\mu\nu}, \quad (3)$$

where

$$S_\rho{}^{\mu\nu} = \frac{1}{2} (K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\lambda\nu}{}_\lambda - \delta_\rho^\nu T^{\lambda\mu}{}_\lambda), \quad (4)$$

and $K^{\mu\nu}{}_\rho$ is contorsion tensor,

$$K^{\mu\nu}{}_\rho = -\frac{1}{2} (T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}). \quad (5)$$

The action of $f(T)$ theory is given by [8, 9, 37]

$$\mathcal{S} = \int d^4x \left[\frac{|e| f(T)}{16\pi G} + \mathcal{L}_M \right], \quad (6)$$

where G is the gravitational constant, $|e|$ is the determinant of vierbein $e^a{}_\mu$, $f(T)$ is a function of torsion scalar T , and \mathcal{L}_M is the Lagrangian of matter. Note that it reduces to TEGR if $f(T) = T$. We obtain the equations of motion for the vierbein from the variation of the action (6) with respect to $e^a{}_\nu$ [8, 9, 37],

$$|e|^{-1} \partial_\mu (|e| S_a{}^{\mu\nu}) f_T(T) + \frac{1}{4} e_a{}^\nu f(T) + S_a{}^{\mu\nu} \partial_\mu T f_{TT}(T) + T^\rho{}_{\mu a} S_\rho{}^{\nu\mu} f_T(T) = 4\pi G e_a{}^\mu \mathcal{T}_\mu{}^\nu, \quad (7)$$

where $S_a{}^{\mu\nu} = S_\rho{}^{\mu\nu} e_a{}^\rho$, $T^\rho{}_{\mu a} = T^\rho{}_{\mu\nu} e_a{}^\nu$, a subscript T denotes a derivative with respect to T , and $\mathcal{T}_\mu{}^\nu = -|e|^{-1} e^a{}_\mu (\delta \mathcal{L}_M / \delta e^a{}_\nu)$ is the energy-momentum tensor of matter. In this work, we consider the perfect fluid, and its energy-momentum tensor is given by $\mathcal{T}^{\mu\nu} = -p g^{\mu\nu} + (p + \rho) u^\mu u^\nu$, where ρ , p and u are the energy density, pressure and velocity four-vector, respectively, with $u^0 = 1$ and $u^i = 0$. In the literature (e.g. [8, 9, 29]), some typical forms of $f(T)$ have been extensively considered, for instance,

$$f(T) = T + c_1 (-T)^n, \quad (8)$$

and

$$f(T) = T + c_2 \left(1 - e^{-c_3 \sqrt{-T}} \right), \quad (9)$$

where c_1 , c_2 , c_3 and n are constants. As is shown in the literature, these two specific $f(T)$ models are interesting and viable. For example, the constraints on these two typical $f(T)$ models were considered in e.g. [41, 42]; the equation of state in these two $f(T)$ theories was studied in e.g. [43]; the Noether and Hojman symmetries were considered in e.g. [44]; the dynamical behavior was studied in e.g. [45], and so on. In the literature, there are many relevant works on these two typical $f(T)$ models, and it is hard to mention them in details. We refer to e.g. [29] (and the references therein) for a comprehensive review on $f(T)$ theory. We stress that our results are not only valid for the specific forms of $f(T)$ given in Eqs. (8) and (9). In fact, the results obtained in the present work do not rely on the specific forms of $f(T)$, and they are valid for all $f(T)$ theories satisfying $f(T=0) = 0$ and $f_T(T=0) \neq 0$, if the adiabatic approximation and the weak field limit are considered (see below).

B. Weak field limit of $f(T)$ theory

Here, we consider the weak field limit of $f(T)$ theory. Following e.g. [21, 28], we restrict ourselves to the adiabatic approximation, in which the evolution of the universe is very slow in comparison with local dynamics. It means that we can choose the Minkowski metric η_{ab} instead of Friedmann-Robertson-Walker (FRW) metric as the background metric [28]. In fact, the quantitative studies (e.g. [30–32]) showed that the physics of gravitationally bound systems (such as galaxies, clusters, or planetary systems) which

are small compared to the radius of curvature of the cosmological background is essentially unaffected by the expansion of the universe. Here, we briefly mention the key points of these quantitative studies (e.g. [30–32]). If the gravitationally bound system is massive enough, the local spacetime might be affected, and hence it should take an interpolating metric. By calculating the effective potential minimum, one can find the numerical evolution of the radius of the gravitationally bound system. It is found that the effect of cosmic expansion on the gravitationally bound systems on (or below) the galactic/cluster scales are insignificant. We refer to e.g. [30–32] for details. Thus, it is reasonable to adopt the adiabatic approximation and use the Minkowski instead of FRW metric as the background metric in the present work. In this case, the vierbein $e^a{}_\nu$ can be rewritten as a sum of flat component and small perturbed component, namely

$$e^a{}_\nu = \text{diag}(1 + \Phi, 1 - \Psi, 1 - \Psi, 1 - \Psi), \quad (10)$$

where Φ and Ψ are the gravitational potentials ($\Phi \ll 1$, $\Psi \ll 1$), and they are functions of spacetime coordinates. From Eq. (1), it is easy to get the metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \text{diag}(1 + 2\Phi, -1 + 2\Psi, -1 + 2\Psi, -1 + 2\Psi). \quad (11)$$

Note that $f(T)$ gravity is not a local Lorentz invariant theory [46] (we thank the referee for pointing out this issue). In general, the lack of local Lorentz symmetry implies that there is no freedom to fix any of the components of the tetrad [46]. The choice of tetrad is crucial, and different tetrads will lead to different field equations [50]. The authors of [50] suggested to speak of a “good” tetrad if it imposes no restrictions on the form of $f(T)$. As mentioned above, we adopt the adiabatic approximation and use the Minkowski metric as the background metric. Note that for the Minkowski metric $\text{diag}(+1, -1, -1, -1)$, the simplest choice for the background vierbein is given by $\text{diag}(+1, +1, +1, +1)$, according to Eq. (1). This motivates us to suggest the perturbed vierbein in Eq (10), which is actually the simplest background vierbein $\text{diag}(+1, +1, +1, +1)$ plus small perturbed components Φ , Ψ . In fact, the tetrad choice (10) is “good” according to the standard proposed in [50]. On the other hand, the metric $g_{\mu\nu}$ in Eq (11) obtained from the tetrad choice (10) via Eq. (1) coincides with the well-known one considered in Newtonian gravity and the weak field limit of GR (see e.g. [6, 7, 28] and the references therein). This also supports the tetrad choice given in Eq. (10). In fact, the role played by the non-local Lorentz invariance in perturbation theory in the context of $f(T)$ gravity is far from clear, and the literature is quite confusing about this. However, in our particular case the Minkowski metric is perturbed at the first order, and the lack of local Lorentz invariance could not be a problem (we thank the referee for pointing out this issue). Anyway, let us move forward.

Substituting Eq. (10) into Eq. (7), we find the linearized vierbein field equations as

$$E_0{}^0 = -\partial_i \partial^i \Psi f_{T|0} = 4\pi G \rho, \quad (12)$$

$$E_0{}^i = \partial_0 \partial^i \Psi f_{T|0} = 0, \quad (13)$$

$$E_i{}^0 = \partial_j \partial^0 \Psi \delta_i^j f_{T|0} = 0, \quad (14)$$

$$E_i{}^j = \frac{1}{2} \partial_k \partial^j (\Psi - \Phi) \delta_i^k f_{T|0} = 4\pi G \partial_i \partial^j \pi^S \quad (i \neq j), \quad (15)$$

$$E_i{}^j = \left[-\delta_i^j \partial_0 \partial^0 \Psi + \frac{1}{2} \partial_k \partial^j (\Psi - \Phi) \delta_i^k - \frac{1}{2} \partial_k \partial^k (\Psi - \Phi) \delta_i^j \right] f_{T|0} = 4\pi G p \quad (i = j), \quad (16)$$

where the indices 0 and $i, j, k = 1, 2, 3$ indicate the time component and space components of the spacetime coordinates, respectively; ρ and p are the density and pressure of the perfect fluid, respectively; π^S is the scalar component of the anisotropic stress [34]; ∂_k denotes $\partial/\partial x^k$; the subscript “|0” means that the corresponding quantity is evaluated at $T = 0$, for instance, $f_{T|0} = f_T(T = 0)$. Note that the evaluation at $T = 0$ in the dynamical equations is a consequence of the linearization process. We have adopted the adiabatic approximation, and hence the Minkowski spacetime is in the vacuum, i.e. the zeroth order approximation of the energy-momentum tensor is vanishing ($\bar{\rho} = \bar{p} = 0$). This makes the right-hand sides of Eqs. (13) and (14) be zero, while ρ, p are treated as the perturbations of the same order of Φ, Ψ . On the other hand, one can find $f(T = 0) = 0$ by considering the zeroth order approximation

of the vierbein field equations (namely the unperturbed equations). Actually, this is also a consequence of the adiabatic approximation (so the zeroth order approximation of the energy-momentum tensor is vanishing). In fact, the most popular $f(T)$ theories in the literature, e.g. the ones given by Eqs. (8) and (9), naturally satisfy $f(T = 0) = 0$. This also justifies the adiabatic approximation used in this work. From Eq. (12), if $\rho \neq 0$, we see that $f_{T|0} \neq 0$ and the gravitational potential Ψ must depend on the space coordinates (namely it cannot depend only on the time). Then, from Eqs. (13) and (14), we find that $\partial_0 \Psi = 0$, and hence Ψ must depend only on the space coordinates. If one adopts the zero-anisotropic-stress assumption, it is easy to see that $\Phi = \Psi$ and $p = 0$ from Eqs. (15) and (16). However, in the general cases we need not assume the vanishing of the anisotropic stress, and then $\Phi = \Psi$ and $p = 0$ are not necessary. In the following discussions, we only use the Poisson's equation given in Eq. (12), without the assumptions $\Phi = \Psi$ and $p = 0$. The Poisson's equation of gravitational potential in $f(T)$ theory (namely Eq. (12)) can be rewritten as

$$\Delta \Psi \equiv \nabla^2 \Psi = 4\pi G \alpha \rho, \quad (17)$$

where $\Delta \equiv \nabla^2$ is the well-known Laplacian, and

$$\alpha \equiv 1/f_{T|0}. \quad (18)$$

As mentioned above, the results obtained in the present work do not rely on the specific forms of $f(T)$, and they are valid for all $f(T)$ theories satisfying $f(T = 0) = 0$ and $f_T(T = 0) \neq 0$, if the adiabatic approximation and the weak field limit are considered. The information of the function $f(T)$ is mainly encoded in the parameter α given in Eq. (18). If $f(T) = T$, or equivalently $\alpha = 1$, our results reduce to the ones of GR. At first glance, one might redefine the gravitational constant as $G_{\text{eff}} = \alpha G$, and then the Poisson's equation (17) becomes the one of GR. However, we note that this does not work if $\alpha \rightarrow 0$ or even $\alpha < 0$, otherwise the effective gravitational constant becomes zero or negative. In fact, as is shown in Sec. V, if $\alpha \leq 0$, the disks are unconditionally stable. This is significantly different from the cases of $\alpha > 0$ (especially $\alpha = 1$ in GR), in which the disks can be stable only when some conditions are satisfied. Thus, we do not redefine the gravitational constant, so that the cases of $\alpha \leq 0$ can be included, which can lead to interesting results.

III. DISPERSION RELATION

In this section, we study the behavior of density waves in self-gravitating differentially rotating disks in $f(T)$ theory by using density wave theory [21, 23]. As mentioned in Sec. I, to the best of our knowledge, Lin and Shu [23] are the pioneers who developed density wave theory for the first time. For convenience, in the present work, we mainly follow the modern formalism of density wave theory given in e.g. [21]. We firstly calculate the gravitational potential by using the Poisson's equation (17), and then determine how the potential perturbation affects the equilibria of the self-gravitating system. We finally obtain the dispersion relations for gaseous and stellar systems in $f(T)$ theory, respectively. Note that the stability will be analyzed in the next section (Sec. IV).

Since the gravitational force is long-range, galaxy can be regarded as a strongly coupled system. In principle, the wave patterns should be determined numerically. To be simple, in our analysis, we assume that the spiral galactic disks are tightly wound. In density wave theory, the long-range coupling can be neglected for tightly wound density waves [21, 23]. Therefore, it is reasonable to adopt the tight-winding approximation (also known as Wentzel-Kramers-Brillouin (WKB) approximation in the literature) to make the analytic derivations possible and simple. As mentioned in e.g. [21, 28], the WKB approximation can be used in most cases without much loss of generality.

The stability analysis in self-gravitating fluid (gaseous) system is similar to the analysis in stellar system, because a fluid system is supported against gravity by gradients in the pressure p , while a stellar system is supported by gradients in the stress tensor [21]. However, the stellar disk is more complicated than gaseous disk. It is easy to use the results of gaseous disk to analyze the difference between various theories. Since the response of a gaseous disk exhibits most of the important features that also occur in the stellar case, we find that it is helpful to consider spiral structure in gaseous disks before we tackle the more complicated stellar disks.

A. Potential of a tightly wound spiral pattern

Now, we consider an extremely thin disk whose surface density is Σ , occupying the plane $z = 0$ and rotating in the x and y directions (we use the Cartesian coordinate system (x, y, z) here). So, the Poisson's equation (17) can be rewritten as

$$\Delta\Psi = 4\pi G\alpha\Sigma\delta(z). \quad (19)$$

The properties of an equilibrium system are described by the time-independent quantities. For the Poisson's equation, we consider small perturbations of potential Ψ and surface density Σ , which can be written as $\Psi = \Psi_0 + \Psi_1$, $\Sigma = \Sigma_0 + \Sigma_1$, where the subscript “0” denotes unperturbed (time-independent) components and the subscript “1” denotes perturbed components. Then, the linearized Poisson's equation reads

$$\Delta\Psi_1 = 4\pi G\alpha\Sigma_1\delta(z). \quad (20)$$

To solve Eq. (20), it is convenient to adopt the following ansatz $\Sigma_1 = \Sigma_a \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, $\Psi_1(z=0) = \Psi_a \exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, where Σ_a and Ψ_a are constants, ω is the frequency and \mathbf{k} is the wave vector. Note that the perturbations should be real. If the complex functions Σ_1 and $\Psi_1(z=0)$ satisfy Eq. (20), $\Re(\Sigma_1)$ and $\Re(\Psi_1(z=0))$ also satisfy the equation, where $\Re(\xi)$ gives the real part of the complex function ξ . So, we allow Σ_1 and $\Psi_1(z=0)$ to be complex, with the understanding that the physical density is given by its real part instead. We refer to [21] for detailed discussions.

Without loss of generality, we choose x -axis to be parallel to \mathbf{k} , so $\mathbf{k} = k\mathbf{e}_x$, where k is the wavenumber and \mathbf{e}_x is the unit vector parallel to x -axis. Since when $z \neq 0$ we have $\Delta\Psi_1 = 0$, the potential perturbation can be rewritten as

$$\Psi_1 = \Psi_a e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t) - |kz|}. \quad (21)$$

To relate Ψ_a to Σ_a , we integrate Eq. (20) from $z = -\zeta$ to $z = +\zeta$, where $\zeta > 0$ and $\zeta \rightarrow 0$. Since $\partial^2\Psi_1/\partial x^2$ and $\partial^2\Psi_1/\partial y^2$ are continuous at $z = 0$, but $\partial^2\Psi_1/\partial z^2$ is not, we have

$$\lim_{\zeta \rightarrow 0} \int_{-\zeta}^{+\zeta} \frac{\partial^2\Psi_1}{\partial z^2} dz = \lim_{\zeta \rightarrow 0} \left. \frac{\partial\Psi_1}{\partial z} \right|_{-\zeta}^{+\zeta} = -2|k|\Psi_1 = \lim_{\zeta \rightarrow 0} 4\pi G\alpha\Sigma_1 \int_{-\zeta}^{+\zeta} \delta(z) dz = 4\pi G\alpha\Sigma_1, \quad (22)$$

where we have used Eq. (21) in the second “=”. So, it is easy to find that

$$\Psi_a = -\frac{2\pi G\alpha}{|k|} \Sigma_a. \quad (23)$$

If $\alpha \rightarrow 1$, Eq. (23) reduces to the Newtonian case as expected. It is worth noting that the derivation of Eq. (23) only involves the fundamental aspects of wave equations, regardless of whether the disks are uniformly rotating or differentially rotating. So, Eq. (23) is valid for both the uniformly and differentially rotating disks [21]. Note also that Cartesian coordinates are used when we derive the gravitational potentials. One should be aware that we might switch to polar or cylindrical coordinates in the following derivations, although the transformation between different coordinate systems is easy.

The disks considered above are the uniformly rotating disks in which we can use general wave dynamics. If we instead consider the differentially rotating disks, the structure of spiral arms in the disks should be regarded as density waves, according to the density wave theory [21].

To be self-contained, here we briefly introduce the so-called tightly wound spiral arms [21]. The spiral arm of galaxy can be viewed as a spiral curve which can be written as $\phi + g(R, t) = \text{const.}$, where ϕ is the azimuthal angle, R is the distance from the center of galaxy to the point of the curve (we use the polar coordinate system here). We assume that the galaxy with m -fold symmetry has $m > 0$ spiral arms (m is a positive integer). So, the locations of all m arms are given by $m\phi + F(R, t) = \text{const.}$, where $F(R, t) = mg(R, t)$ is the shape function. The condition for tight winding is related to the so-called pitch angle ϑ of the arm which is given by $\cot \vartheta = |kR/m|$ at any radius r . When $\cot \vartheta = |kR/m| \gg 1$, namely the pitch angle ϑ is very small, we can say that the spiral arms are tightly wound.

The surface density of a zero-thickness differentially rotating disk is the sum of an unperturbed component $\Sigma_0(R)$ and a perturbed component $\Sigma_1(R, \phi, t)$. For a tightly wound spiral, according to density

wave theory [21], $\Sigma_1(R, \phi, t)$ can be expressed as a form separating the rapid variations in density as one passes between arms from the slower variation in the strength of the spiral pattern as one moves along an arm. It is convenient to write the perturbed surface density Σ_1 as

$$\Sigma_1 = H(R, t) e^{i(m\phi + F(R, t))}, \quad (24)$$

where $H(R, t)$ is a smooth function of radius R that gives the amplitude of the spiral galaxy. Since the surface density oscillates rapidly around zero mean, the perturbed potential at a given location will be determined by the properties of the pattern within a few wavelengths of that location. We can replace the shape function $F(R, t)$ by its Taylor series in the vicinity of (R_0, ϕ_0) , $F(R, t) \rightarrow F(R_0, t) + k(R_0, t)(R - R_0)$. The perturbed surface density Σ_1 can be rewritten as

$$\Sigma_1 \simeq \Sigma_a e^{ik(R_0, t)(R - R_0)}, \quad (25)$$

where

$$\Sigma_a = H(R_0, t) e^{i(m\phi_0 + F(R_0, t))}. \quad (26)$$

From Eq. (25), we find that the spiral wave closely resembles a plane wave with wave vector $\mathbf{k} = k\mathbf{e}_R$ in the vicinity of (R_0, ϕ_0) , where \mathbf{e}_R is the unit vector parallel to R direction. Note that the gravitational potential for uniformly rotating disks is given by Eq. (23). So, we can write the similar perturbed potential in differentially rotating disks as

$$\Psi_1 \simeq \Psi_a e^{ik(R_0, t)(R - R_0)}, \quad (27)$$

where Ψ_a is given by Eq. (23). We refer to e.g. [21] and references therein for a detailed introduction of density wave theory.

B. Gaseous disks

The stability analysis in self-gravitating fluid system is similar to the analysis in stellar system, because a fluid system is supported against gravity by gradients in the pressure p , while a stellar system is supported by gradients in the stress tensor [21]. On the other hand, the stability analysis of fluid system has been studied previously in the literature. So, it is better to consider the response of tightly wound density waves of self-gravitating gaseous system at first, and then deal with the similar analysis of self-gravitating stellar system later.

For our aim, it is necessary to obtain the corresponding continuity equation and the Euler equation, which are two of the most important equations in fluid dynamics and are useful for the stability analysis of the gaseous system. We consider a zero-thickness disk occupying the plane $z = 0$ (we use the cylindrical coordinate system (R, ϕ, z) here). In Newtonian gravity, the continuity equation and Euler equation are given by [21]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (28)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Psi, \quad (29)$$

respectively, where the pressure p and velocity \mathbf{v} of the perfect fluid act only in the disk plane $z = 0$, and the volume density ρ can be replaced by the surface density Σ for our aim. According to [28], since only the potential Ψ appears in the equation of motion of a test particle, the continuity equation and Euler equation in $f(T)$ theory take the same forms in Newtonian gravity or $f(R)$ theory, i.e. Eqs. (28) and (29), while the information of $f(T)$ theory is carried by the modified gravitational potential Ψ .

By using cylindrical coordinates, the continuity equation (28) can be rewritten as

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma R v_R) + \frac{1}{R} \frac{\partial}{\partial \phi} (\Sigma R v_\phi) = 0, \quad (30)$$

where v_R and v_ϕ are the radial and azimuthal components of velocity, respectively. On the other hand, the radial and azimuthal components of Euler equation (29) can be rewritten as

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial \Psi}{\partial R} - \frac{1}{\Sigma} \frac{\partial p}{\partial R}, \quad (31)$$

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R} \frac{\partial \Psi}{\partial R} - \frac{1}{\Sigma R} \frac{\partial p}{\partial \phi}. \quad (32)$$

Since the gaseous disk is introduced to serve as a heuristic model of stellar system, we are free to choose a simple equation of state $p = K\Sigma^\gamma$, where K and γ are constants. It is convenient to define a specific enthalpy $h \equiv \int \Sigma^{-1} dp = \frac{\gamma}{\gamma-1} K\Sigma^{\gamma-1}$, and then Eqs. (31) and (32) can be simplified to

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_R}{\partial \phi} - \frac{v_\phi^2}{R} = -\frac{\partial}{\partial R} (\Psi + h), \quad (33)$$

$$\frac{\partial v_\phi}{\partial t} + v_R \frac{\partial v_\phi}{\partial R} + \frac{v_\phi}{R} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_R}{R} = -\frac{1}{R} \frac{\partial}{\partial R} (\Psi + h). \quad (34)$$

As mentioned above, the equilibrium of system is determined by the time-independent quantities. For gaseous system, we can consider that the spiral wave is a small perturbation on axisymmetric disk. We can write surface density $\Sigma = \Sigma_0 + \Sigma_1$, radial component of velocity $v_R = v_{R0} + v_{R1} = v_{R1}$, azimuthal component of velocity $v_\phi = v_{\phi 0} + v_{\phi 1}$, gravitational potential $\Psi = \Psi_0 + \Psi_1$ and enthalpy $h = h_0 + h_1$, where the subscript “0” denotes unperturbed quantities and the subscript “1” denotes small perturbed quantities. Note that in the equilibrium, we have $v_{R0} = 0$ and $\partial \Psi_0 / \partial \phi = \partial h_0 / \partial \phi = 0$. The linearized equations of Eqs. (30), (33) and (34) are given by

$$\frac{\partial \Sigma_1}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} (\Sigma_0 R v_{R1}) + \Omega \frac{\partial \Sigma_1}{\partial \phi} + \frac{\Sigma_0}{R} \frac{\partial v_{\phi 1}}{\partial \phi} = 0, \quad (35)$$

$$\frac{\partial v_{R1}}{\partial t} + \Omega \frac{\partial v_{R1}}{\partial \phi} - 2\Omega v_{\phi 1} = -\frac{\partial}{\partial R} (\Psi_1 + h_1), \quad (36)$$

$$\frac{\partial v_{\phi 1}}{\partial t} + \Omega \frac{\partial v_{\phi 1}}{\partial \phi} + \left[\frac{d(\Omega R)}{dR} + \Omega \right] v_{R1} = -\frac{1}{R} \frac{\partial}{\partial \phi} (\Psi_1 + h_1), \quad (37)$$

where $\Omega = v_{\phi 0} / R$ is the circular frequency. For differentially rotating disk, Ω is related to radial coordinate R . According to the similar analysis in Sec. III A, the ansatz for the solutions of Eqs. (35), (36) and (37) can be written as

$$\Psi_1 = \Re \left[\Psi_a(R) e^{i(m\phi - \omega t)} \right], \quad (38)$$

$$\Sigma_1 = \Re \left[\Sigma_a(R) e^{i(m\phi - \omega t)} \right], \quad (39)$$

$$v_{R1} = \Re \left[v_{Ra}(R) e^{i(m\phi - \omega t)} \right], \quad (40)$$

$$v_{\phi 1} = \Re \left[v_{\phi a}(R) e^{i(m\phi - \omega t)} \right], \quad (41)$$

$$h_1 = \Re \left[h_a(R) e^{i(m\phi - \omega t)} \right], \quad (42)$$

where $m > 0$. Substituting Eqs. (38)–(42) into Eqs. (36) and (37), we get

$$v_{Ra} = -\frac{i}{\Theta} \left[(m\Omega - \omega) \frac{d}{dR} (\Psi_a + h_a) + \frac{2m\Omega}{R} (\Psi_a + h_a) \right], \quad (43)$$

$$v_{\phi a} = \frac{1}{\Theta} \left[-2B \frac{d}{dR} (\Psi_a + h_a) + \frac{m(m\Omega - \omega)}{R} (\Psi_a + h_a) \right], \quad (44)$$

where

$$B = -\frac{1}{2} \left[\frac{d(\Omega R)}{dR} + \Omega \right] = -\Omega - \frac{R}{2} \frac{d\Omega}{dR}, \quad (45)$$

$$\Theta = \kappa^2 - (m\Omega - \omega)^2, \quad (46)$$

in which the epicyclic frequency κ is defined by

$$\kappa^2 = R \frac{d\Omega^2}{dR} + 4\Omega^2 = -4B\Omega. \quad (47)$$

The linearized equation of state is given by

$$h_a = \gamma K \Sigma_a \Sigma_0^{\gamma-2} = v_s^2 \Sigma_a / \Sigma_0, \quad (48)$$

where the sound speed v_s is given by $v_s^2 = \gamma K \Sigma_0^{\gamma-1}$.

In principle, the linearized continuity equation (35), the expressions of velocity Eqs. (43), (44), and the linearized equation of state Eq. (48) can be numerically solved to yield the global forms of the self-consistent density waves in a given disk [33]. Instead, here we use the WKB approximation (namely $|kR| \gg m \geq 1$) mentioned above to get the analytic local solutions for density waves. The potential of a tightly wound wave can be written as

$$\Psi_a(R) = W(R) e^{iF(R)} = W(R) \exp \left(i \int_0^R k dR \right), \quad (49)$$

where $k = dF(R)/dR$ and $|kR| \gg 1$. From Eqs. (20), (38), (39) and (48), we see that Σ_a and h_a share the same factor $\exp(iF(R))$ with Ψ_a . Therefore, we can write $d(\Psi_a + h_a)/dR = ik(\Psi_a + h_a)$ and then neglect the terms proportional to $(\Psi_a + h_a)/R$ in Eqs. (43) and (44) because $|kR| \gg 1$. So, Eqs. (43) and (44) become

$$v_{Ra} = \frac{k(m\Omega - \omega)(\Psi_a + h_a)}{\Theta}, \quad (50)$$

$$v_{\phi a} = -\frac{2iBk(\Psi_a + h_a)}{\Theta}. \quad (51)$$

For the similar reason, the continuity equation (35) can be rewritten as

$$(m\Omega - \omega) \Sigma_a + k \Sigma_0 v_{Ra} = 0. \quad (52)$$

Using Eqs. (52), (50), (48), (46) and (23), we find that the dispersion relation for a gaseous disk in the tight-winding limit is given by

$$(m\Omega - \omega)^2 = k^2 v_s^2 - 2\pi G |k| \alpha \Sigma_0 + \kappa^2. \quad (53)$$

Note that our result is suited for differentially rotating disks. If $\kappa = 2\Omega$, noting Eqs. (47), (45) and $\Omega = v_{\phi 0}/R$, we see that Eq. (53) reduces to the dispersion relation for uniformly rotating disks. On the other hand, if $\alpha \rightarrow 1$, namely $f_{T|0} \rightarrow 1$, this dispersion relation in $f(T)$ theory reduces to the one in Newtonian gravity [21]. Since in the weak field limit GR reduces to Newtonian gravity, the dispersion relation in $f(T)$ theory also reduces to the one in GR when $\alpha \rightarrow 1$. This is not surprising, because $f(T)$ theory is equivalent to GR if $f(T) = T$, as is well known. It is worth noting that the corresponding dispersion relation in $f(R)$ theory [28] is quite different from the one in $f(T)$ theory, namely Eq. (53).

C. Stellar disks

In this subsection, we study the dispersion relation for stellar disks. As a fluid system, the dynamics of gaseous disk is determined by the continuity equation and Euler equation. However, the dynamics

of stellar disk is instead described by the so-called collisionless Boltzmann equation. The collisionless Boltzmann equation in Newtonian gravity (the weak field limit of GR) reads

$$\frac{\partial D}{\partial t} + \mathbf{v} \cdot \nabla D - \nabla \Psi \cdot \frac{\partial D}{\partial \mathbf{v}} = 0, \quad (54)$$

where \mathbf{v} is the stellar velocity and D is the distribution function. As mentioned above, the stability analysis in stellar system is closely similar to the one in fluid system, because a fluid system is supported against gravity by gradients in the pressure p , while a stellar system is supported by gradients in the stress tensor [21]. So, we could use the similar technique used in the analysis for gaseous disks to obtain the dispersion relation for stellar disks. We refer to [21] for more detailed discussions.

The important step is to determine the perturbation \bar{v}_{R1} induced by Ψ_1 in the mean radial velocity of the stars at a given point (R, ϕ) . If the unperturbed orbits are circular, namely the disk is quite “cold”, \bar{v}_{R1} is given by

$$\bar{v}_{Ra} = \frac{m\Omega - \omega}{\Theta} k \Psi_a, \quad (55)$$

which could be obtained from Eq. (50) with $h_a = 0$, since the disk would be dynamically identical to a gaseous disk with zero pressure [21]. This formula is accurate unless the stellar epicycle amplitude is much smaller than the wavelength $2\pi/k$ of spiral pattern. Otherwise, Eq. (55) should be modified to

$$\bar{v}_{Ra} = \frac{m\Omega - \omega}{\Theta} k \Psi_a \mathcal{F}, \quad (56)$$

where $\mathcal{F} \leq 1$ is the reduction factor. For a given potential perturbation, the reduction factor describes how much the response to a spiral perturbation is reduced below the value for a cold disk. Then, we can calculate the response density Σ_a once we have \bar{v}_{Ra} , since the so-called Jeans equation, $\partial \nu / \partial t + \partial(\nu \bar{v}_i) / \partial x_i = 0$, is identical to the continuity equation of the gaseous disk. So, the analogue of Eq. (52) is given by

$$(m\Omega - \omega)\Sigma_a + k\Sigma_0 \bar{v}_{Ra} = 0. \quad (57)$$

According to [21], the reduction factor \mathcal{F} in Newtonian gravity is given by

$$\mathcal{F}(s, \chi) = \frac{1 - s^2}{\sin(\pi s)} \int_0^\pi e^{-\chi(1 + \cos \tau)} \sin(s\tau) \sin \tau d\tau, \quad (58)$$

where

$$s = \frac{\omega - m\Omega}{\kappa}, \quad (59)$$

$$\chi = \left(\frac{k\sigma_R}{\kappa} \right)^2, \quad (60)$$

in which σ_R is the radial velocity dispersion. We refer to e.g. [21] and references therein for a detailed introduction to the reduction factor \mathcal{F} . Note that \mathcal{F} takes the same form in different gravity theories, because the gravitational potential does not appear in Eq. (58). Using Eqs. (56), (57), (58), (46) and (23), we obtain the dispersion relation for stellar disks,

$$(m\Omega - \omega)^2 = \kappa^2 - 2\pi G |k| \alpha \Sigma_0 \mathcal{F}(s, \chi). \quad (61)$$

Again, if $\alpha \rightarrow 1$, namely $f_{T|0} \rightarrow 1$, this dispersion relation in $f(T)$ theory reduces to the one in Newtonian gravity (the weak field limit of GR) [21]. It is worth noting that the corresponding dispersion relation in $f(R)$ theory [28] is quite different from the one in $f(T)$ theory, namely Eq. (61).

The dispersion relations Eqs. (53) and (61) are the main equations for studying the density waves in disks. Although the WKB approximation is valid conditionally, it can be used in most cases without much loss of generality, as mentioned in e.g. [21, 28]. These dispersion relations could provide an invaluable

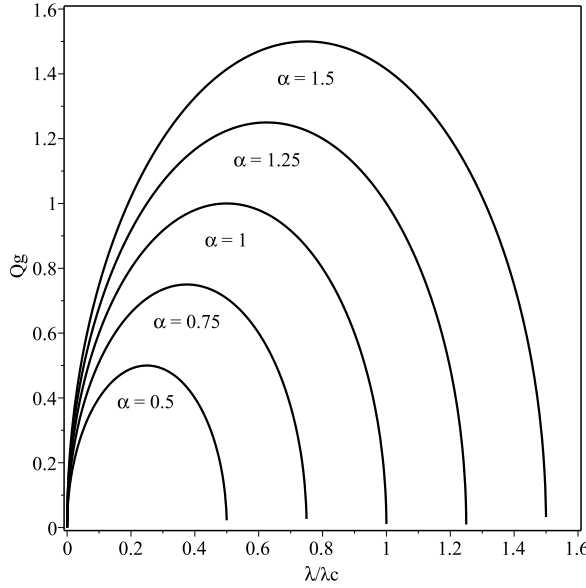


FIG. 1: The neutral stability curves for various α on the $Q_g - \lambda/\lambda_c$ plane according to Eq. (65) obtained in $f(T)$ theory for gaseous disk. Note that the curve with $\alpha = 1$ corresponds to the one in GR and Newtonian gravity. See the text for details.

guide for the numerical analysis of stability, and they establish the relation between wavenumber and frequency that is satisfied by a traveling wave as it propagates across the disk [21].

Comparing the dispersion relations in $f(R)$ theory [28] and $f(T)$ theory for both gaseous and stellar disks, we find that the main difference between them is that the dispersion relations in $f(T)$ theory only relate to the first order derivative of $f(T)$, while the dispersion relations in $f(R)$ theory relate to not only the first but also the second order derivatives of $f(R)$. So, we suppose that it might be a possible hint to distinguish $f(T)$ theory from $f(R)$ theory.

IV. LOCAL STABILITY

In the previous section, we obtained the dispersion relations Eqs. (53) and (61) for gaseous and stellar disks in $f(T)$ theory, respectively. In this section, we apply these relations to determine whether a given disk is locally stable to axisymmetric perturbations. Note that the above analysis is mainly for tightly wound non-axisymmetric disturbances, namely $|kR/m| \gg 1$ and $m > 0$. However, the results obtained above are still valid for the axisymmetric disturbances ($m = 0$), so long as $|kR| \gg 1$ [21]. In the following, we try to obtain the local stability criteria in $f(T)$ theory for both gaseous and stellar disks.

A. Gaseous disks

At first, we consider the local stability of gaseous disks. For $m = 0$, Eq. (53) becomes

$$\omega^2 = k^2 v_s^2 - 2\pi G |k| \alpha \Sigma_0 + \kappa^2. \quad (62)$$

The right-hand side of Eq. (62) is real, so ω^2 should be real. If $\omega^2 > 0$, the frequency ω is real and hence the disk is stable (nb. Eqs. (38)–(42)). On the other hand, if $\omega^2 < 0$, the disk is unstable. The neutral stability curve ($\omega^2 = 0$) is given by

$$k^2 v_s^2 - 2\pi G |k| \alpha \Sigma_0 + \kappa^2 = 0, \quad (63)$$

which is a quadratic equation with respect to $|k|$ in fact. Noting that Eq. (62) can be regarded as a parabola opening upward on the $\omega^2 - |k|$ plane, if there is no real solution for the quadratic equation

with respect to $|k|$ given in Eq. (63), we have $\omega^2 > 0$ for any $|k|$, and hence the gaseous disk is stable against all axisymmetric perturbations. Thus, by requiring that there is no real solution for Eq. (63), it is easy to obtain the condition for axisymmetric stability,

$$Q_g \equiv \frac{v_s \kappa}{\pi G \Sigma_0} > \alpha. \quad (64)$$

This is the local stability criterion in $f(T)$ theory for gaseous disk. In order to plot the curve of neutral stability given in Eq. (63), it is useful to introduce the longest unstable wavelength $\lambda_c = 4\pi^2 G \Sigma_0 / \kappa^2$ and $y = \lambda / \lambda_c$, where $\lambda = 2\pi / |k|$. So, we can recast Eq. (63) as

$$Q_g = 2\sqrt{\alpha y - y^2}, \quad (65)$$

where $Q_g \equiv v_s \kappa / (\pi G \Sigma_0)$ is defined in Eq. (64). In Fig. 1, we plot the neutral stability curves for various α on the $Q_g - \lambda / \lambda_c$ plane according to Eq. (65). Noting that Eq. (62) can be rewritten as

$$Q_g^2 - 4(\alpha y - y^2) = \omega^2 \left(\frac{\kappa}{\pi G |k| \Sigma_0} \right)^2, \quad (66)$$

the disk is stable/unstable in the regions above/below the neutral stability curve for a given α on the $Q_g - \lambda / \lambda_c$ plane, respectively. Note that Eq. (64) reduces to the one in Newtonian gravity if $\alpha \rightarrow 1$.

B. Stellar disks

Let us turn to the local stability of stellar disks. In the case of axisymmetric perturbations ($m = 0$), Eq. (61) becomes

$$\omega^2 = \kappa^2 - 2\pi G |k| \alpha \Sigma_0 \mathcal{F} \left(\omega / \kappa, (k \sigma_R / \kappa)^2 \right). \quad (67)$$

Again, noting Eqs. (38)–(42), if $\omega^2 > 0$ the disk is stable, and if $\omega^2 < 0$ the disk is unstable. The neutral stability curve ($\omega^2 = 0$) is given by

$$\kappa^2 - 2\pi G |k| \alpha \Sigma_0 \mathcal{F} \left(0, (k \sigma_R / \kappa)^2 \right) = 0. \quad (68)$$

By using Eqs. (58)–(60) and the identity $\exp(z \cos \theta) = \sum_{n=-\infty}^{+\infty} I_n(z) \cos(n\theta)$ [21], it can be recast as

$$\frac{|k| \sigma_R^2}{2\pi G \alpha \Sigma_0} = 1 - \exp \left(-\frac{k^2 \sigma_R^2}{\kappa^2} \right) I_0 \left(\frac{k^2 \sigma_R^2}{\kappa^2} \right), \quad (69)$$

where $I_0(\chi)$ is the modified Bessel function. If there is no real solution for Eqs. (68) or (69) with respect to $|k|$, the stellar disk is stable against all axisymmetric perturbations. Note that Eqs. (67)–(69) are closely similar to the ones in Newtonian gravity, except that an additional factor α appears. Therefore, in analogy to the derivations in Newtonian gravity [21], one can obtain the modified local stability criterion in $f(T)$ theory as

$$Q_s \equiv \frac{\kappa \sigma_R}{3.36 G \Sigma_0} > \alpha. \quad (70)$$

Obviously, it reduces to the standard Toomre's criterion in Newtonian gravity if $\alpha \rightarrow 1$.

Note that the neutral stability curve determined by Eqs. (68) or (69) involves an integration function \mathcal{F} given in Eq. (58) or a modified Bessel function I_0 . So, it is not easy to solve the complicated equations given in Eqs. (68) or (69). Some complicated tricks are needed, as in e.g. [28]. However, the resulted curves are similar to the ones in Fig. 1 (see e.g. [28]), and hence we do not present them here.

It is of interest to briefly compare the stability criteria in $f(T)$ theory (Eqs. (64) and (70)) with the ones in $f(R)$ theory obtained in [28]. The main difference is that the stability criteria in $f(T)$ theory involve only the first order derivative of the function $f(T)$ (encoded in the parameter α), whereas the stability criteria in $f(R)$ theory involve not only the first but also the second order derivatives of the function $f(R)$ (encoded in the parameters α and β [28], respectively). This is mainly due to the fact that the equations of motion in $f(T)$ theory are 2nd order, whereas the ones of $f(R)$ theory are 4th order.

V. CONCLUDING REMARKS

In this work, we consider the stability of self-gravitating differentially rotating disks in $f(T)$ theory. At first, we get the Poisson's equation in the weak field limit, using the adiabatic approximation. Then, we obtain the gravitational potential in differentially rotating disks for a small perturbation. By studying the behavior of density wave, we obtain the dispersion relations for both gaseous and stellar disks. Finally, we study the local stability of both gaseous and stellar disks by applying the dispersion relations, and obtain the modified Toomre's criteria (the local stability criteria) in $f(T)$ theory, i.e. Eqs. (64) and (70).

Comparing the local stability criteria in $f(R)$ theory [28] and $f(T)$ theory, we find that the main difference between them is that the local stability criteria in $f(T)$ theory only relate to the first order derivative of $f(T)$, while the local stability criteria in $f(R)$ theory relate to not only the first but also the second order derivatives of $f(R)$. So, we suppose that it might be a possible hint to distinguish $f(T)$ theory from $f(R)$ theory.

Let us observe the local stability criteria (64) and (70) closely. If $\alpha \rightarrow 1$, namely $f_{T|0} \rightarrow 1$, both of Eqs. (64) and (70) reduce to the standard Toomre's criteria in Newtonian gravity. This is not surprising, because $f(T)$ theory is equivalent to GR if $f(T) = T$, while GR reduces to Newtonian gravity in the weak field limit. However, if $\alpha \neq 1$, Toomre's criterion should be modified. If α is larger/smaller than 1, a larger/smaller Q_g (compared with the one in Newtonian gravity or the weak field limit of GR) is needed to make the disk stable, respectively. This means that the disk needs larger/smaller pressure (compared with the one in Newtonian gravity or the weak field limit of GR) to resist the gravitational collapse [21, 28]. We consider that this might be potentially used to distinguish $f(T)$ theory from GR observationally.

It is interesting to consider the case of $\alpha \rightarrow 0$, namely $f_{T|0} \rightarrow \infty$. For instance, the typical form of $f(T)$ in Eq. (9) which has been extensively considered in the literature, can satisfy not only $f(T=0) = 0$ (required by the adiabatic approximation as mentioned in Sec. IIB), but also $f_{T|0} = f_T(T=0) \rightarrow \infty$. In the case of $\alpha \rightarrow 0$ (namely $f_{T|0} \rightarrow \infty$), noting Eqs. (62) and (67), it is easy to see that $\omega^2 > 0$ holds unconditionally, and hence the disks are always stable. This can also be found by considering the local stability criteria (64) and (70), which are satisfied unconditionally if $\alpha \rightarrow 0$. Therefore, the disks are unconditionally stable in e.g. the $f(T)$ theory given in Eq. (9). Similarly, the same arguments also hold in the case of $\alpha < 0$, namely $f_{T|0} < 0$. For instance, $f(T) = T + c_1(1 - \exp(c_2T))$ can satisfy $f(T=0) = 0$ (required by the adiabatic approximation), and $f_{T|0} < 0$ if the model parameters satisfy $c_1c_2 > 1$. The disks are also unconditionally stable. So, the kinds of $f(T)$ theories with $\alpha \leq 0$ might be observationally tested on (or below) the galactic scales.

Note that in our analysis, we restrict ourselves to the adiabatic approximation, and hence the Minkowski metric can be taken as the background metric. It means that the evolution of the universe is very slow in comparison with local dynamics, and the physics of gravitationally bound systems (such as galaxies, clusters, or planetary systems) which are small compared to the radius of curvature of the cosmological background is essentially unaffected by the expansion of the universe. The adiabatic approximation has been supported by many quantitative studies (e.g. [30–32]) since 1945 at least. However, it has been challenged recently, and the debate is not completely settled by now. On the other hand, as mentioned in Sec. IIB, $f(T=0) = 0$ is required by the adiabatic approximation. However, not all $f(T)$ theories satisfy this requirement in fact. Based on the above two arguments, it is also plausible to give up the adiabatic approximation, and choose e.g. the FRW metric as the background metric. In this case, the requirement $f(T=0) = 0$ is not necessary. However, as mentioned in e.g. [34], another constraint $f_{TT|0} = f_{TT}(T=0) = 0$ can be imposed by the requirement of no anisotropic stress (of course, this constraint can also be given up, by allowing anisotropic stress). On the other hand, since the background metric is chosen to be e.g. the FRW metric or the interpolating metric [32, 35] in this case, all the derivations in this work will become fairly complicated, and hence the modified local stability criteria should also become complicated accordingly. We also stress that the tetrad selection and the perturbative treatment could become significantly cumbersome. We leave this issue as an open question.

Another important assumption used in this work is the WKB approximation (namely the tight-winding approximation). As mentioned above, a natural criterion for the validity of the WKB approximation for axisymmetric waves is $|kR| \gg 1$, which actually corresponds to $\lambda/R \ll 2\pi$. In fact, most of the numerical experiments require $\lambda/R \lesssim 2$ for the accuracy of the WKB results. So, the WKB approximation is valid in the solar neighborhood at least, and can be used in many galactic disks. However, the WKB

approximation cannot be applied to the loose wound spiral structures. In this case, there are no analytic methods to determine the stability of a given disk to small perturbations, and the numerical methods should be used instead.

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